Möbius strip segmented into flat trapezoids: Design-build Project to represent the two departments of Architecture and Mathematics of the Technische Universität Darmstadt

1. Introduction: Möbius strip as trade fair stand

The Möbius strip trade fair stand is part of a series of 3 trade fair stands that were designed and built by students of the Architecture faculty of Darmstadt Technical University. This project series of 1:1 design and build projects focused on two main issues: firstly to introduce computer-driven parametric design and production strategies, techniques and technologies into architectural education and secondly to not just design a project, but to actually build and operate it.

Parametric design and production enables architects to construct their designs from elements that are neither completely identical nor totally different - as was dictated by the principles of serialized mass production. Instead, elements can be similar and varied like members of a family. This means more complexity and diversification within the built structures and promises more appropriate designs that can respond better to the specifics individualities of users, program, site etc.

Most educational projects in architecture schools never leave the drawing board, so to speak. While this is understandable with projects of large sizes like large cultural buildings or...
residential complexes, it excludes from architectural education the very important development a design project undergoes as it is built: ideas have to adapt and change according to the restraints and possibilities of budget, availability of material, manufacturing and building machines and issues of transportation. In this development, quite different skills are needed than the conceptualising and presentation skills necessary for designing. Instead, organization, management, craftsmanship and aptitude with materials and tools become essential.

The annual Hobit fair of the Technical University Darmstadt, where prospective students become acquainted with the university and its faculties and departments, provided a perfect occasion to conduct a course that merged our two educational ambitions: the architecture faculty needed a fair stand that would provide information about the courses and should also through its sculptural qualities communicate what the study of architecture is about.

The Möbius strip was chosen as basis for the stand’s shape because it in itself displays fascinating aspects of geometry, and because constructing it on the scale of a fair stand requires geometrical and constructive effort that would again in themselves illustrate important issues of the study of architecture. Moreover, the faculty of mathematics had expressed its interest to collaborate on the stand, and the Möbius strip is one of the most iconical and well-known mathematical objects. The collaboration between the two faculties would also become an important part of the exhibition as collaboration with different disciplines is an important part of architectural work.

We chose to segment the Möbius strip into several flat trapezoids because it would provide us with different possibilities of use: it could become a counter, a shelf, a table and other things. As it turned out, the segmentation was much more difficult than expected.

![Figure 2: transformation of Möbius strip to segmentation into flat trapezoids.](image)

![Figure 3: flat trapezoids provide different possibilities of use](image)
2. Mathematical problem: constructing a Möbius strip out of regular flat trapezoids

We introduced the topic of Möbius strips through a talk from a mathematical point of view. Its focus, though, lay not in a discussion of topological issues - an area that would usually be opened up through such a start. Instead, we focused on geometrical issues, partly because of the task at hand, and partly because past experiences with students of architecture have shown that projective geometry let alone higher dimensional aspects are not a very suitable topic for them.

This time, though, a higher dimensional model came into being naturally from the very beginning. The final mathematical way to see the problem can be written as follows. We prepare a rectangular paper strip with endpoints A, B, C, and D, say in counter-clockwise order. We identify or we glue line segment AB with line segment CD, such that A equals C and such that B equals D. The resulting surface with boundary becomes non-orientable, we obtain a Möbius strip. Now we draw a regular zig-zag with end points on the boundary BC and AD of the Möbius strip such that the strip will be partitioned into isosceles triangles. We next use a remaining smaller strip by cutting off another strip parallel along the Möbius strip’s boundary. Thus from each triangle we are left with a trapezoid that we than require to be of flat shape. The edges of adjacent trapezoids are considered to function as hinges. Of course, we require to have these trapezoids without self-intersections.

Mathematical questions immediately arose that were close to what the architects were interested in. Under which conditions does such a model exist? There is e.g. a triangulation of the Möbius strip by Ulrich Brehm that cannot be realized with flat triangles and without self-intersections. Of course, the students in our „trapezoidulation problem“ already had a lot of models showing their existence, didn’t they? However, when it comes to calculate the angles between adjacent trapezoids, we cannot trust the paper versions, can we? When the trapezoids become in a certain limit rectangles, we are sure that a solution cannot exist. And what happens near such a rectangular limit case?

Let us assume that we have n trapezoids and therefore n angles between adjacent ones. Mathematicians introduce for each angle between adjacent trapezoids a dimension and they describe the problem as an n-dimensional one. The conditions A=C and B=D reduce the dimension n of the realization space by five, three conditions for the three coordinates of the requirement A=C and only two for the condition B=D because the distance from A to B equals the distance from C to D. The resulting manifold of dimension n-5 that describes our solutions in this n-dimensional space is not connected. We have to assume in general that there are solutions with knots. A knot solution cannot be transformed continuously into an unknotted version. Has the reader noticed that n has to be odd and that we cannot choose n to be very small? Any suggested solution of a polyhedral Möbius strip of the architects with
certain angles measured from their models is very unlikely to be a correct point on our (n-5)-
dimensional solution manifold for polyhedral Möbius strips. Can we use nevertheless such
an „angle point’’ of the architects in n-space to find in the vicinity of it a true solution on the
solution manifold? Can we somehow project this point onto this manifold? It was good for
the architects to have some mathematical advice for realizing their 1:1 model.
The following formulation might tell the non-mathematician what the problem looked like
for a mathematician. We assume for that for simplicity that the exact solution manifold is a
curve in space, i.e., a one dimensional manifold. In other words, only points on this curve
can be accepted for a model. The students than suggested only „points” near this curve and
the mathematical problem was to correct their data in order to get an acceptable solution
point near the given one on the solution curve.

3. Designing and constructing the trade fair stand

The students had three months to design, manufacture and build the stand.
We started with a design competition that produced a large number of alternative designs,
all based on Möbius strips, and all accommodating different situations of use, like information
desk and various display possibilities. One of these alternatives was selected by a jury for its
use of parametrics and its promise in terms of buildability.

The students then worked as a group to develop the winning project further. Physical models
were produced and tested as to their qualities as fair stands. The parametrics in this design
stage consisted of varying the number and precise positioning of trapezoids to produce dif-
ferent shapes and possibilities for use. An important issue in producing and discussing these
studies and their qualities consisted of making implicit factors explicit. Architects very often
unconsciously set themselves a set of implicit rules that govern their designs but which are
also often broken, leading to much confusion. We urged the students to make their implicit
rules explicit so that their being broken woud surface and their productive qualities be dis-
cussed.
To be able to actually build the Möbius strip, the trapezoids were further broken up into
straight ribs. This was to resolve the complex 3D joint of the trapezoids, and the ribs pro-
vided additional means of display. The ribs were to be manufactured of glass and wood. The
joints between the different strips or parts of the original trapezoids were designed adjustable
so that the geometry of the Möbius strip closing in on itself could be realized on site.
Without a mathematical solution or procedure, segmenting the strip geometry into flat
trapezoids became very difficult and cumbersome as the last neighbouring trapezoids never
matched up. The collaboration with mathematicians proved extremely fruitful as they provided the architects with an interactive software application that could generate many variations of perfectly segmented strips and export them in a file format that the architects’ CAD packages were able to use (see below).

4. An interactive application that offers solutions to the mathematical/geometrical problem

Without an attempt to explain the mathematical background of the problem that we tried to describe in Section 2, the mathematical help for the architects was provided by a mathematical student Sebastian Stammler. He used the mathematical software package „MATLAB“s built-in optimization software and wrote corresponding software parts in close connection with the architects. From Wikipedia we have the following description: „MATLAB (matrix laboratory) is a numerical computing environment and fourth-generation programming language. Developed by MathWorks, MATLAB allows matrix manipulations, plotting of functions and data, implementation of algorithms, creation of user interfaces, and interfacing with programs written in other languages. “

The set of angles between adjacent trapezoids served as the input of the software. An animation visualized how the starting strip could be moved to its final position as a polyhedral Möbius strip. This helped a lot to imagine how the precise solution would finally look like.

![Figure 6: examples of configurations of Möbius strips with varying number of trapezoids](image)

Furthermore, all spatial information for manufacturing the trapezoids were produced and the interfaces for other production software were provided. It was possible to play with parameters like the number of trapezoids and the shape of the trapezoids defined via its angles or the width of the strip.

![Figure 7: the interactive MATLAB programme provides different segmentations](image)
5. Building and operating the trade fair stand

Building the trade fair stand with students necessitates the limitation of the involved crafts to a number of 3 or 4. Furthermore, since the university does not provide CNC tools or CAAM machinery that can produce objects larger than simple models, the students had to find regional companies that would provide machinery and material.

The two-dimensional segments made from plexiglass and timber were cut by a local joiner’s CNC laser cut machine. The three-dimensional spacing elements between the ribs were cut from a simple PVC pipe. The specific geometry for the spacers was produced by 5-dimensional CNC milling. The data for both elements was generated in Rhino. The transfer to the lasercut was handled through AutoCAD; the transfer to the milling machine through the mechanical engineering file-format Parasolid.
Whenever possible, the students operated the machines themselves and handled transportation and storage of the construction components. The assembly of the parts was tested twice: first before manufacturing the components, in a small-scale model. And a second time a few days before the opening of the fair in a kind of dress rehearsal at the architecture faculty over the weekend. Both tests proved invaluable and provided information that influenced both the construction details and the final assembly procedure on site.
After the final assembly in the spaces of the fair, the students took shifts to operate the stand, to explain its construction and principles to the visiting high-school students.

6. Conclusion

The idea of the design of the Möbius strip consisted not only of one specific object: the design being parametrized meant that a whole family of related objects was designed. Together with a fully integrated digital production chain, this would in principle allow for the production of a large number of fair stands, all stemming from the same design, but adjusted to situational specifics like place, size, arrangement of different functions.

Realizing design-build projects in a non-CAAM environment constitutes a big challenge for all participants: knowledge of different software packages and readiness to quickly familiarize with unknown data formats, flexibility in the time schedule and disposition to transport the material to the machinery.

The project not only introduced the students to the possibilities of designing with parametric modeling software, but also familiarized them with the initial difficulties and the following large payback in employing them for actual production. The finished stands went far beyond what they initially had believed possible.

Employing the Möbius strip geometry proved indeed to communicate issues of geometry and the fascination of solving geometrical and technical problems and challenges. As we had hoped, the fair stand in itself became a showcase for architecture and mathematics. Although the stand contained a number of different exhibition objects, the Möbius strip geometry became an exhibit in itself.
Figure 13: lighting installation of Möbius strip.

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figure 1 top and bottom right, figure 9, figure 10 right, figure 13 by Stefan Daub

figure 1 bottom left
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References

Movie showing segmented Möbius strip

Movie showing interactive programme

Movie showing set-up of the built Möbius strip